ON THE STRUCTURE OF THE FIELD OF MICROSTRESSES OF STEADY PLASTIC FLOW

(O STRUKTURE POLIA MIKRONAPRIAZHENII RAZVITOGO PLASTICHESKOGO TECHENIIA)

PMM Vol.28, № 2, 1964, pp. 326-334

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(Received December 20, 1963)

In all models of a solid body that are considered in continuum mechanics (elastic body, ideal plastic body, etc.), it is only in the very simple case of a homogeneous and isotropic body that the most detailed analysis is undertaken. Actual solid bodies that can be assumed homogeneous and isotropic with respect to their macroscopic properties have in the majority of cases an inhomogeneous microscopic structure, the characteristic dimension of which greatly exceeds the molecular one. A typical example is provided by polycristals, in which the size of individual grains is of the order of 10^{-5} to 10^{-1} cm. This size is small in comparison with the usual size of objects made of polycrystals; however, it exceeds the characteristic molecular scale by several powers of ten. Polymers, as was recently proved by V.A. Kargin and T. I. Sogolova, also have an inhomogeneous structure of super-molecular dimensions.

The parameters of the individual micro-inhomogenieties (geometric characteristics, elastic properties, structure, etc.) can be treated as random quantities. Therefore, the characteristics of the state of stress in such bodies (stresses, and others) are random functions of time and position in the body. For the description of the stress-deformation state of bodies with an arbitrary deviation from linear elasticity, it is not only the mean values — the mathematical expectations — of the considered quantities that are of significance, but also their correlation functions. In fact, for cases that can be treated with the aid of the linear equations of the usual theory of elasticity, it follows by virtue of the linearity of these equations that the correlations drop out of the avereged equations. Therefore, it is possible to obtain a closed system of linear equations for the averaged quantities and to study independently the linear system of equations for the correlation function, if they are of interest on their own. For nonlinear processes, in particular plastic flow, this does not in general apply. The equations for the averaged values contain correlation moments which represent an essential element of the macroscopic investigation.

There is an analogy between plastic flow and turbulance in a viscous fluid, which has been pointed out, in lucid form by Taylor [1]. This analogy follows from both the nonlinear character of these phenomena and also from their dissipative properties. The nonlinearity leads to a redistribution of energy among the various degrees of freedom of motion, and the dissipation of energy on a small scale eliminates the possibility of its accumulation. A definite advance in the theory of turbulence has been achieved by means of statistical methods, and the clearest results can be obtained for the simplest particular case of homogeneous, isotropic turbulence. Apart from a certain very particular form of flow, the turbulent field is not so simple. Nevertheless, as was first established by Kolmogorov [2 and 4], for stationary turbulent flow the assumption of isotropy and of homogeneity results in motions of sufficiently small scale which are independent of the structure of the established field (local homogeneity and local isotropy of a stationary turbulent flow). This property, together with the hypothesis of transmission of energy by a cascade of motions of different scales, enabled Kolmogorov to describe turbulent motions of appropriate scale factors. The above-mentioed analogy makes it possible to carry out the corresponding examination in the theory of steady-state plastic flow also.

1. Statistical character of a field of microstresses. Pysical solids have an inhomogeneous microstructure of supermolecular dimensions. For example, polycrystalline materials consist of an aggregate of crystals of different shapes, sizes, orientations, boundary states, etc. Essentially, we have the same picture for mountain rocks and polymers. It is natural that geometrically identical samples made up from various pieces of a macro-homogeneous material are not identical microscopically, and that the stresses produced in these samples under the action of one and the same loading will also be different. For a physical body, even the representation of the complete structure of the inhomogeneity presents insurmountable difficulties, not to mention the mathematical difficulties of calculating the states of stress and strain in a body consisting of a large number of crystals. However, the random character of the micro-inhomogeneities make it possible to apply statistical methods of macroscopic description. Thus, statistical methods have been used in the paper of Lifshits and Rozentsveig [5] to give a discription of the elastic properties of polycrystals.

Correspondingly, we will regard the totality of geometrically identical samples taken from different parts of a macro-homogeneous and isotropic material as a statistical ensemble. The average over the ensemble of the value of all kinds of characteristics (the mathematical expectation)* will be denoted by the symbol $\langle\,\rangle$, and hence the stress $\sigma_{ij}(\mathbf{x})$ at a given point $\mathbf{x}$ of the sample can be representd in the form

$$\sigma_{ij}(\mathbf{x}) = \langle\sigma_{ij}(\mathbf{x})\rangle + \sigma_{ij}'(\mathbf{x}) \tag{1.1}$$

where $\sigma_{ij}(\mathbf{x})$ is the pulsation of the stress or the microstress**. The microstresses represent a random tensor field with mean values equal to zero. The basic characteristic of this field is the correlation tensor of fourth order

$$R_{ijkl}(\mathbf{x}, \mathbf{x}+\mathbf{r}) = \langle\sigma_{ij}'(\mathbf{x})\,\sigma_{kl}'(\mathbf{x}+\mathbf{r})\rangle \tag{1.2}$$

which characterizes the connection of the stresses at different points. The scale factor of distances at which the values of the components of the correlation tensor become negligibly small compared with their values when $r = 0$

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* In view of the ergodic hypothesis, the average over the ensemble can be replaced by either a spatial or temporal mean.

** This terminology is a little different from the conventional.

will be called the scale factor of the microstress correlation.

Because of the symmetry of tensor σ_{ij}' and definition (1.2), we have

$$R_{ijkl} = R_{jikl} = R_{jilk}, \qquad R_{ijkl}(\mathbf{x}, \mathbf{x}+\mathbf{r}) = R_{klij}(\mathbf{x}, \mathbf{x}-\mathbf{r}) \tag{1.3}$$

The field of stresses satisfies the equilibrium equations

$$\frac{\partial \sigma_{\alpha k}(\mathbf{x}+\mathbf{r})}{\partial x_\alpha} = \frac{\partial \sigma_{\alpha k}(\mathbf{x}+\mathbf{r})}{\partial r_\alpha} = 0 \tag{1.4}$$

Averaging (1.4) and subtracting the averaged equation from (1.4), we obtain

$$\frac{\partial \langle \sigma_{\alpha k}(\mathbf{x})\rangle}{\partial x_\alpha} = 0, \qquad \frac{\partial \sigma_{\alpha k}'(\mathbf{x})}{\partial x_\alpha} = 0 \tag{1.5}$$

Multiplying the second equation (1.4) by $\sigma_{ij}'(\mathbf{x})$ and averaging, we obtain an equation for the correlation tensor

$$\frac{\partial R_{ijk\alpha}(\mathbf{x}, \mathbf{x}+\mathbf{r})}{\partial r_\alpha} = 0 \tag{1.6}$$

We note that the first invariants of the correlation tensor R_{ijkl} which are obtained by pairwise contraction of the indices, have definite physical meaning. In fact, isolating the spherical part of the stress tensor, we have

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{1.7}$$

Substituting this into (1.2) and contracting, we find

$$R_{\alpha\beta\alpha\beta}(\mathbf{x}, \mathbf{x}+\mathbf{r}) = 3P(\mathbf{x}, \mathbf{x}+\mathbf{r}) + T(\mathbf{x}, \mathbf{x}+\mathbf{r}), \quad R_{\alpha\alpha\beta\beta}(\mathbf{x}, \mathbf{x}+\mathbf{r}) = 9P(\mathbf{x}, \mathbf{x}+\mathbf{r}) \tag{1.8}$$

where

$$P(\mathbf{x}, \mathbf{r}) = \langle p'(\mathbf{x})\, p'(\mathbf{x}+\mathbf{r})\rangle, \qquad T(\mathbf{x}, \mathbf{x}+\mathbf{r}) = \langle \tau_{\alpha\beta}'(\mathbf{x})\, \tau_{\alpha\beta}'(\mathbf{x}+\mathbf{r})\rangle$$

The elastic energy per unit mass of a body with shear modulus μ and bulk modulus K can be expressed as

$$W = \frac{1}{4\mu\rho}\tau_{\alpha\beta}^2 + \frac{1}{2K\rho}p^2 = W_0 + W' = \left\{\frac{1}{4\mu\rho}\langle\tau_{\alpha\beta}\rangle^2 + \frac{1}{2K\rho}\langle p\rangle^2\right\} + $$
$$+ \left\{\frac{1}{4\mu\rho}\tau_{\alpha\beta}'^2 + \frac{1}{2K\rho}p'^2\right\} + \left\{\frac{1}{2\mu\rho}\langle\tau_{\alpha\beta}\rangle\,\tau_{\alpha\beta}' + \frac{1}{K\rho}\langle p\rangle p'\right\} \tag{1.9}$$

where the term W_0 in the first brackets is the sum of the elastic energy of change of shape and of volumetric dilatation for the averaged stresses, the term in the second brackets is the analogous quantity for the microstresses, and the third brackets contain terms expressing the interaction between the averaged stresses and the microstresses (in a plastic body W is the energy released as a result of complete unloading). From the expression for the mean energy the last terms drop out, so that the mean energy of the microstresses can be expressed in the form

$$\langle W'\rangle = \frac{1}{4\mu\rho}P(\mathbf{x}, \mathbf{x}) + \frac{1}{2K\rho}T(\mathbf{x}, \mathbf{x}) \tag{1.10}$$

Thus, $R_{\alpha\alpha\beta\beta}(\mathbf{x}, \mathbf{x})$ and $R_{\alpha\beta\alpha\beta}(\mathbf{x}, \mathbf{x})$ can be expressed in the form of linear combinations of the mean elastic energies of change of shape and of the volumetric deformation for the microstresses.

As has already been pointed out, for sufficiently small stresses, when all the deformations are reversible and the behavior of the bodies can be described by the classical linear equation of the theory of elasticity, the determination of the tensor of the mean stresses and of the correlation tensor can be carried out separately. This problem was treated by Volkov [6]. Any divergence from classical, linear elasticity leads to nonlinearity of the basic relations, and the system of equations for the mean stresses ceases to be closed. In particular, this occurs in plastic flow.

2. Homogeneous, isotropic microstresses. The simplest possible assumption about the structure of the field of microstresses is the assumption that it is homogeneous and isotropic. This assumption implies, in particular, the homogeneity and isotropy of the correlation tensor (1.2).

An isotropic field of microstresses can be produced, for example, by subjecting a sphere to repeated, identical extensions or compressions in different random directions (by rolling it between rigid plane surfaces). The general expression for the homogeneous, isotropic tensor that satisfies relations (1.3) has the form

$$R_{ijkl}(\mathbf{r}) = A(r)\, n_i n_j n_k n_l + B(r)\,(n_i n_j \delta_{kl} + n_k n_l \delta_{ij}) + C(r)\,(n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} + n_j n_l \delta_{ik}) + D(r)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + E(r)\,\delta_{ij}\delta_{kl} \qquad (2.1)$$

where $r = |\mathbf{r}|$, $\mathbf{n}\,(n_i) = \mathbf{r}/r$; and A, B, C, D, E are scalar functions of r. On substituting (2.1) into equation (1.6) we obtain the three independent equations

$$\begin{aligned} &2A + rA' - 2B + rB' - 4C + 2rC' = 0 \\ &2B + rB' + 2C + rE' = 0 \\ &B + 3C + rC' + rD' = 0 \end{aligned} \qquad (2.2)$$

In place of the functions A, B, C, D, E it is possible to consider linear combinations of them which have direct physical meaning

$$\begin{aligned} &f_1(r) = n_i n_j n_k n_l R_{ijkl} = A + 2B + 4C + 2D + E \\ &f_2(r) = n_i n_j t_k t_l R_{ijkl} = B + E, \quad f_3(r) = n_i t_j n_k t_l R_{ijkl} = C + D \qquad (2.3) \\ &f_4(r) = \tau_i t_j \tau_k t_l R_{ijkl} = D, \quad f_5(r) = \tau_i \tau_j t_k t_l R_{ijkl} = E \\ &(\mathbf{n} = \mathbf{r}/r, \ \mathbf{n}^2 = \mathbf{t}^2 = \boldsymbol{\tau}^2 = 1, \ \mathbf{nt} = \mathbf{n}\boldsymbol{\tau} = \mathbf{t}\boldsymbol{\tau} = 0) \end{aligned}$$

Equations (2.2) assume the form

$$\begin{aligned} {}^1\!/_2 r f_1' &= -f_1 + f_2 + 2f_3 \\ {}^1\!/_2 r f_2' &= -f_2 - f_3 + f_4 + f_5 \qquad (2.4) \\ r f_3' &= -f_2 - 3f_3 + 3f_4 + f_5 \end{aligned}$$

Corresponding to the functions $f_1, \ldots, f_5$, it is possible to introduce integral scale factors for the correlations in the form

$$L_i = \int_0^\infty \frac{f_i(r)dr}{f_i(0)} \tag{2.5}$$

Thus, by virtue of the equilibrium equations, only two correlation moments remain independent.

In the following investigation it will sometimes be convenient to pass over to Fourier transforms. A random field of microstresses can be resolved into a Fourier-Stieltjes integral

$$\sigma_{ij}'(\mathbf{x}) = \int e^{i\mathbf{kx}} dZ_{ij}(\mathbf{k}) \tag{2.6}$$

Substituting this expression into the definition of the correlation tensor (1.2), we obtain

$$R_{ijkl}(\mathbf{x}+\mathbf{r}, \mathbf{x}) = \iint e^{i[\mathbf{kr}+(\mathbf{k}-\mathbf{k}')\mathbf{x}]} \langle dZ_{ij}(\mathbf{k})\, dZ_{kl}(\mathbf{k}')\rangle \tag{2.7}$$

(Here the bar indicates the complex conjugate quantity).

For the homogeneity of the field (independence of $\mathbf{x}$) it is necessary that

$$\langle dZ_{ij}(\mathbf{k})\, d\overline{Z}_{kl}(\mathbf{k}')\rangle = \Phi_{ijkl}(\mathbf{k})\, \delta(\mathbf{k}-\mathbf{k}')\, d^3\mathbf{k} d^3\mathbf{k}' \tag{2.8}$$

i.e. that the spectral components of the fields of microstresses should be noncorrelative. Thus, for a homogeneous field we obtain

$$R_{ijkl}(\mathbf{r}) = \int \Phi_{ijkl}(\mathbf{k})\, e^{i\mathbf{kr}}\, d^3\mathbf{k} \tag{2.9}$$

The Fourier transform Φ_{ijkl} of the correlation tensor can be defined as the inverse of (2.9) so that

$$\Phi_{ijkl}(\mathbf{k}) = \frac{1}{8\pi^3}\int R_{ijkl}(\mathbf{r})\, e^{-i\mathbf{kr}}\, d^3\mathbf{r} \tag{2.10}$$

It satisfies the symmetry relations

$$\Phi_{ijkl} = \Phi_{jikl} = \Phi_{ijlk}, \qquad \Phi_{ijkl}(\mathbf{k}) = \Phi_{klij}(-\mathbf{k}) = \overline{\Phi}_{klij}(\mathbf{k}) \tag{2.11}$$

Moreover, the form $\Phi = \Phi_{ijkl}(\mathbf{k})\, X_i Y_j \overline{X}_l \overline{Y}_k$ is positive-definite for arbitrary complex vectors X, Y. Equations (1.6) can be written down in the form

$$k_i \Phi_{ijkl} = 0 \tag{2.12}$$

Then the general expression of the isotropic spectral tensor Φ_{ijkl} can be written down in the form

$$\Phi_{ijkl} = a(k)\, \varkappa_{ij}\varkappa_{kl} + b(k)(\varkappa_{ik}\varkappa_{jl} + \varkappa_{jk}\varkappa_{il}) \quad (\varkappa_{ij} = k^{-2}i(k_i k_j - k^2\delta_{ij})) \tag{2.13}$$

where $a(k)$ and $b(k)$ are functions of the magnitude of the wave vector.

The coefficients $a(k)$ and $b(k)$ can be expressed in terms of quantities having direct physical meaning. For this we make use of the spectral representation introduced in Section 1 of the functions P and T in the case of a homogeneous, isotropic field of microstresses. We have

$$P(r) = \int \Pi(\mathbf{k})\, e^{i\mathbf{kr}} d^3\mathbf{k} = 4\pi \int_0^\infty \Pi(k)\, k^2 \frac{\sin kr}{kr}\, dk$$

$$T(r) = 4\pi \int \Theta(k)\, k^2 \frac{\sin kr}{kr}\, dk \tag{2.14}$$

The integration over the angle can be carried out separately by virtue of the isotropy.

On the ground of (1.8), (2.9) and (2.13) we have

$$\begin{aligned} \Phi_{\alpha\beta\alpha\beta}(k) &= 3\Pi(k) + \Theta(k) = 2a(k) + 6b(k) \\ \Phi_{\alpha\alpha\beta\beta}(k) &= 9\Pi(k) = 4a(k) + 4b(k) \end{aligned} \tag{2.15}$$

In agreement with (1.10) and (2.14), for a homogeneous, isotropic field of microstresses we obtain

$$\langle W' \rangle = \int_0^\infty \frac{4\pi}{\mu\rho} E(k)\, dk + \int_0^\infty \frac{16\pi}{3K\rho} e(k)\, dk$$

$$E(k) = {}^1/_4 k^2\, \Theta(k), \quad | \quad e(k) = {}^3/_8 k^2 \Pi(k) \tag{2.16}$$

so that $E(k)$ and $e(k)$, to within an accuracy of a constant dimensional multiplier, are the spectral densities of the energies of change of shape and of volumetric dilatation, respectively.

Expressing $a(k)$ and $b(k)$ directly in terms of $E(k)$ and $e(k)$, we can rewrite (2.13) in the form

$$\Phi_{ijkl}(k) = \frac{E(k) - e(k)}{k^2} \{\varkappa_{ik}\varkappa_{jl} + \varkappa_{jk}\varkappa_{il} - \varkappa_{ij}\varkappa_{kl}\} + \frac{6e(k)}{k^2} \varkappa_{ij}\varkappa_{kl} \tag{2.17}$$

3. Local properties of steady plastic flow. We will consider a body which is in a macro-homogeneous, stress-deformation state and which is composed of microscopically inhomogeneous material. We will assume that the material, in regard to its properties, approximates an ideal plastic body and, further, that the individual elements of the supermolecular structure deform plastically in an almost isotropic manner. In particular, in application to polycrystals this means that the crystallites have a large number of slip planes.

We will follow the course of development of plastic deformations in a body with increase of loading. For definiteness we will speak of polycrystals. At the beginning, when the loading is sufficiently small, the material deforms elastically. Then, on attaining a certain load, the most loaded and the favorably orientated crystallites undergo plastic deformations. If the microinhomogeneities in the material are distributed sufficiently uniformly, the mean distance l_1 between the first plastically deformed crystallites can be very great, in particular, great in comparison with the mean dimensions of a crystallite. Thus, microstresses arise* with the scale factor l_1. With further increase of the load, plastic deformations arise in new crystallites which are then located closer to one another so that there arises a random distribution of microstresses with the scale factor $l_2 < l_1$. With subsequent increase of the load, in the same way there arise microstresses of scale factor $l_3 < l_2 < l_1$, and so forth, so that in steady plastic flow there is an aggregate of microstresses with scale factors from l_1 down to dimensions of the order of the mean grain size d and smaller.

For bodies that are almost ideally plastic, yielding occurs in a narrow interval of load variations. That the scale factors of microstresses can significantly exceed the grain size has been verified experimentally (see the very important work by Pashkov and Bratukhina [7]). Under the conditions of an inhomogeneous stress-deformation state, the upper scale factor of the microstresses can be determined by the scale factor of the inhomogeneities.

For microstresses of greatest scale factor l_1 we encounter anisotropy of the field of applied loads, so that microstresses of this scale factor cannot be considered as isotropic. In passing to the subsequent, smaller scale factor l_2, this anisotropy becomes smaller, since on the anisotropy of the microstresses of greatest scale factor, a new random element is superimposed. In passing to microstresses of even smaller scale factors, further smoothening of the anisotropy arises, so that it can be assumed that microstresses having a scale factor very much smaller that l_1 are isotropic. We emphasize that isotropy of the microstresses of smallest scale factors is assured by the presence of a large number of slip surfaces on the individual crystallites. The behavior of a material with a small number of slip systems will be significantly affected by the texture that is formed during the deformation process. This texture leads to anisotropic microstresses of small scale factor.

The picture presented shows that in steady plastic flow the field of microstresses with scale factor smaller than a certain l is not only homogeneous and isotropic but also has a definite autonomy, i.e. an independence of details of the structure of the basic state of stress and microstresses with larger scale factors. In other words, in steady plastic flow the field of microstresses possesses a local isotropic and homogeneous property. Thus, for corresponding scale factors the structural tensors [2 to 4], which are made up of the characteristics of the stress-deformation state for steady plastic flow, must be isotropic and homogeneous. In particular, the tensor

$$D_{ijkl}(\mathbf{x},\mathbf{x}+\mathbf{r}) = \langle [\sigma_{ij}(\mathbf{x}) - \sigma_{ij}(\mathbf{x}+\mathbf{r})]\,[\sigma_{kl}(\mathbf{x}) - \sigma_{kl}(\mathbf{x}+\mathbf{r})]\rangle \qquad (3.1)$$

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* By characteristic scale factor we mean, as usual, the order of magnitude of the distances over which the stresses change significantly.

must be homogeneous and isotropic for $r < l$. Hence it follows that Formula (2.1) is also valid for the structural tensor D_{ijkl}. For an explanation of the connections between the coefficients A, B, C, D, E of the structural tensor D_{ijkl} we note that, by virtue of the autonomy of the field of microstresses with the scale factor smaller than l, we can obtain these connections by considering the particular case when there is complete isotropy and homogeneity throughout the whole field of stresses. In this case

$$D_{ijkl} = 2\langle\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x})\rangle - 2\langle\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x}+\mathbf{r})\rangle = \\ = 2\langle\sigma_{1ij}\,\sigma_{1kl}\rangle - 2\langle\sigma_{1ij}\sigma_{2kl}\rangle \tag{3.2}$$

when the indices correspond to point 1 with position vector $\mathbf{x}$ and to point 2 with position vector $\mathbf{x}+\mathbf{r}$. By differentiating the last relation with respect to the coordinates of point 2, we obtain

$$\frac{\partial D_{ijkl}}{\partial x_{2k}} = -2\left\langle\sigma_{1ij}\frac{\partial\sigma_{2kl}}{\partial x_{2k}}\right\rangle = 0 \tag{3.3}$$

since on account of the equilibrium equation $\partial\sigma_{2kl} / \partial x_{2k} = 0$. By virtue of the homogeneity, the last relation can be written out in the form

$$\frac{\partial D_{ijkl}}{\partial r_k} = 0 \tag{3.4}$$

(r_k are the components of $\mathbf{r}$). In exact analogy to the preceding, we find that the functions A, B, ... are connected by the same three differential relations (2.2), and consequently for a complete description of the tensor D_{ijkl} it is sufficient to know two of these functions.

We will now consider the process of transforming the energy in the course of steady plastic flow. The work of the external loads under conditions of developed plastic flow is completely converted into the work of plastic deformations, and this conversion takes place with the plastic deformation of the individual grains. Relying on the above-given pattern of steady plastic flow, the process of conversion of energy can be represented as the transmission of energy from the macroscopic stresses to the microstresses of greatest scale factor, and from these to the stresses of smaller scale factors until there is no dissipation of energy at the expense of plastic deformation of the grains, i.e. in the microstresses with a scale factor l_0 smaller than the mean dimension of the grain.

Similar to the inertial interval introduced by Kolmogorov in the theory of turbulence, it is possible to make an assumption concerning the existence of intervals of scale factors of the microstresses (elastic interval) in which, on the one hand, we have local isotropy and homogeneity and, on the other hand, characteristic scale factors are sufficiently great in comparison with the dimension l_0 such that the microstresses of the considered interval of scale factors are not associated with an appreciable dissipation of energy.

The relations characterizing the plastic flow can be expressed in form of functions of kinematic quantities of the type like the "kinematic stress" σ_{ij}/ρ , of the energy absorbed by plastic strains in unit mass, etc., from the dimension of which has been excluded the mass at the expense of division by the corresponding power of density. In particular, the coefficients A, B, C, D, E, which determine the kinematic structure tensor D_{ijkl}, have under such conditions the dimensions L^4T^{-4}. In the elastic interval all the kinematic characteristics of plastic flow can be determined by only two quantities: the distance r and the energy ε that can be absorbed in a unit mass in unit time. From dimensional considerations we have

$$A = \alpha\,(\varepsilon r)^{4/3}, \quad B = \beta\,(\varepsilon r)^{4/3}, \quad C = \gamma\,(\varepsilon r)^{4/3}$$
$$D = \delta\,(\varepsilon r)^{4/3}, \quad E = \lambda\,(\varepsilon r)^{4/3} \tag{3.5}$$

For the microstresses of all scale factors $r < 1$, the significant parameter will also be that scale factor l_0 of the microstresses in which the dissipation of energy occurs in the plastic deformation. This leads to functions of the form $A = \varepsilon^{4/3}r^{4/3}f_A\,(r\,/\,l_0)$, where f_A is some dimensionless function, and similarly for the other functions.

Substituting (3.5) into (2.2), we obtain the following expressions for the coefficients γ, λ and δ in terms of α and β

$$\gamma = \tfrac{5}{2}\,\alpha - \tfrac{1}{2}\,\beta, \qquad \delta = -\tfrac{65}{8}\,\alpha + \tfrac{7}{8}\,\beta, \qquad \lambda = -\tfrac{15}{4}\,\alpha - \tfrac{7}{4}\,\beta \tag{3.6}$$

There is also interest in giving the spectral formulation of the adopted hypotheses and the results obtained. The first hypothesis consists in the fact that the fields of microstresses with wave numbers $k > l^{-1}$ are homogeneous. Hence there is a lack of correlation of the microstresses in this spectral band of wave numbers, i.e. Formula (2.8). The second hypothesis – the hypothesis of a cascade in the space of wave numbers – consists in the fact that, in the interval $k > l^{-1}$, the characteristics of the microstresses can be determined by the following parameters: $k(L^{-1})$, $\varepsilon(L^2T^{-3})$, and $l_0(L)$. Therefore, the kinematic densities of energy of change of shape and of volumetric dilatation can, by virtue of dimensional considerations, be written in the form

$$E = \varepsilon^{4/3}k^{-7/3}f_E\,(kl_0), \qquad e = \varepsilon^{4/3}k^{-7/3}f_E\,(kl_0) \tag{3.7}$$

The third hypothesis means that in the interval $k > l^{-1}$ there exists a region of wave numbers $k \ll l_0^{-1}$. In this region $kl_0 \ll 1$, so that f_E and f_e become constants and relations (3.7) can be written down in the form

$$E = C_E\,\varepsilon^{4/3}k^{-7/3}, \qquad e = C_e\,\varepsilon^{4/3}k^{-7/3} \tag{3.8}$$

The fourth hypothesis consists in the fact that in the interval $k > l^{-1}$ we have isotropy of the fields of microstresses. By virtue of these hypotheses and Formula (3.8) for the spectral representation Φ_{ijkl} of the correlation tensor, a relation of the type (2.17) results

$$\Phi_{ijkl} = \varepsilon^{4/3}k^{-13/3}[(C_E - C_e)(\varkappa_{ik}\varkappa_{jl} + \varkappa_{jk}\varkappa_{il} - \varkappa_{ij}\varkappa_{kl}) + 6C_e\varkappa_{ij}\varkappa_{kl}] \quad (3.9)$$

The dimensionless constants C_E and C_e are connected by the constants α and β introduced earlier.

Making use of the property of integral transform of the structural tensor [3]

$$D_{ijkl} = 2\int \Phi_{ijkl}(\mathbf{k})(1 - \cos \mathbf{kr})\, d^3\mathbf{r} \quad (3.10)$$

we obtain

$$C_E \approx (0.2\beta - 3\alpha)\, 10^2, \qquad C_e \approx (-0.04\beta - 0.5\alpha)\, 10^2 \quad (3.11)$$

Thus, in the elastic interval of scale factors and of wave numbers it turns out to be easy to obtain the expression for the structural tensor of the field of microstresses and of the spectral representation of the correlation tensor to within an accuracy of two universal constants.

The results obtained once more clearly demonstrate that the idea of local isotropy and homogeneity, as well as the cascade hypothesis introduced by Kolmogorov in the theory of turbulence, have a very general meaning for a wide class of nonlinear distributed (continuous) systems with dissipation.

The authors sincerely thank A.S. Monin for valuable advice and R.L. Salganik for helpful discussion.

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Translated by D.B.McV.